

SO WHAT IS A CONTINUED FRACTION?

- ▮ A c.f. is defined to be an expression of the form

$$b_1 + \frac{a_1}{b_2 + \frac{a_2}{b_3 + \frac{a_3}{b_4 + \frac{a_4}{b_5 + \ddots}}}}$$

- ▮ If all numerator's equal 1, then the expression is considered a simple continued fraction.

- ▮ Note how the denominators (the b_i 's) can be put into a 1-1 correspondence with the Natural Numbers.
- ▮ If this sequence is finite, then the expression is considered a finite simple continued fraction.

Using the addition operator as a counter, if we add consecutive complex fractions we can form another sequence.

The elements (c_i 's) of this sequence are called convergents.

$$b_1 + \frac{a_1}{b_2 + \frac{a_2}{b_3 + \frac{a_3}{b_4 + \frac{a_4}{b_5 + \ddots}}}}$$

$$b_1, \quad b_1 + \frac{a_1}{b_2}, \quad b_1 + \frac{a_1}{b_2 + \frac{a_2}{b_3}}, \quad b_1 + \frac{a_1}{b_2 + \frac{a_2}{b_3 + \frac{a_3}{b_4}}}, \quad \dots$$

0 1 2 3

KEY EQUATION

- While each convergent could be converted to a simple fraction by hand this is tedious.
- Instead I use the following recurrence relation which can be proved by induction:

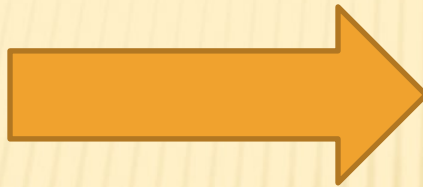
$$c_0 = \frac{A_0}{B_0} = \frac{1}{0}$$

$$c_1 = \frac{A_1}{B_1} = \frac{b_1}{1}$$

$$c_{i+1} = \frac{b_{i+1} \cdot A_i + A_{i-1}}{b_{i+1} \cdot B_i + B_{i-1}}$$

EXAMPLE

$$1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{5}}}}$$



$$c_0 = \frac{A_0}{B_0} = \frac{1}{0}$$

$$c_1 = \frac{A_1}{B_1} = \frac{1}{1}$$

$$c_2 = \frac{2 \cdot 1 + 1}{2 \cdot 1 + 0} = \frac{3}{2}$$

$$c_3 = \frac{3 \cdot 3 + 1}{3 \cdot 2 + 1} = \frac{10}{7}$$

$$c_4 = \frac{4 \cdot 10 + 3}{4 \cdot 7 + 2} = \frac{43}{30}$$

$$c_5 = \frac{5 \cdot 43 + 10}{5 \cdot 30 + 7} = \frac{225}{157}$$

- As (c_i) increases each convergent gives a better approximation to the c.f.
- If (c_i) is finite, then c_n is equal to the c.f.

APPLICATIONS

- ▮ Given a particular c.f. we can now find its convergents in an efficient manner.
- ▮ Most applications arise in two ways:
 - ▮ Applications starting from a given c.f. or sequence
 - ▮ Applications which build a c.f. or sequence from clever algorithms
- ▮ Examples include:
 - ▮ Finding roots to quadratic polynomials
 - ▮ Converting decimals to fractions(both rational and irrational)

STARTING WITH A GIVEN SEQUENCE

$(k, k, k\dots)$



$$x^2 - kx - 1$$



$$x = \frac{k + \sqrt{k^2 + 4}}{2}$$

- ▮ Suppose we have a constant sequence.
- ▮ If we use each element of this sequence as our denominators (the b_i 's) and use our key equation the convergents approach a particular number.
- ▮ It turns out this number is in fact a root to the quadratic equation to the left.

STARTING WITH A QUADRATIC POLYNOMIAL

$$x^2 - kx - 1 = 0$$

$$x - k = \frac{1}{x}$$

$$x = k + \frac{1}{x}$$

$$= k + \frac{1}{k + \frac{1}{k + \frac{1}{k + \ddots}}}$$

- ▮ Divide the equation by x , and then solve for x .
- ▮ This form of the equation is similar to the form of a fixed-point iteration used in numerical analysis.

C.F.'S HAVE COOL PROPERTIES

- ▮ If we use $k=1$ in our formulas we get the well known golden ratio $0.615384615... = \phi$
- ▮ An interesting property of this number is that
$$\Theta = \frac{1}{\phi}$$
- ▮ If we use $k=2$ in our formulas we get $2.414213562... = \xi$
- ▮ An interesting property of this number is that $1/\xi = .414213562...$
- ▮ Thus for every positive integer value of k we can make c.f.'s that limit to numbers with this reciprocal property!

CONVERTING DECIMALS TO FRACTIONS

- ▮ To calculate the b_i in terms of a given decimal d we need the following clever algorithm
- ▮ This algorithm was shameless stolen from (Kennedy, 2009)

$$Z_1 = d$$

$$Z_{i+1} = \frac{1}{Z_i - [Z_i]}$$

$$i = 1, 2, 3, \dots$$

$$b_i = [Z_i]$$

$$Z_1 = d = .12$$

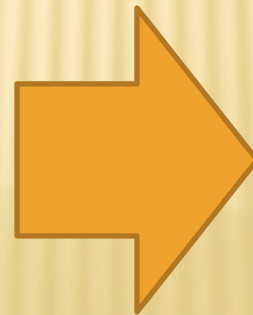
$$Z_2 = \frac{1}{.12 - [.12]} = \frac{1}{.12} = 8.\overline{33}$$

$$Z_2 = \frac{1}{8.\overline{33} - [8.\overline{33}]} = \frac{1}{.\overline{33}} = 3$$

thus,

$$b_1 = 0, b_2 = 8, b_3 = 0$$

□ By our key formula we have



$$0 + \frac{1}{8 + \frac{1}{3}}$$

WORKS CITED

- ▮ *Discovering A Queer Property of Golden Ratio*. (n.d.). Retrieved 12 7, 2010, from Rohit Naringrekar: http://www.rohitn.com/numbers/numbers_Goldenratio.aspx
- ▮ Euler, Leonhard, "An Essay on Continued Fractions" translated by Myra F. Wyman and Bostwick F. Wyman, *Math Systems Theory* 18 (1985) 295-328 (1985)
- ▮ Kennedy, J. (2009, 11 22). *John Kennedy's Downloadable Papers*. Retrieved 12 7, 2010, from John Kennedy's SMC Home Page: http://homepage.smc.edu/kennedy_john/papers.htm
- ▮ Olds, C. (1963). *Continued Fractions*. New York: Random House.